Chosen topics:

I. Tsirelson bound(s).

II. The Tsirelson problem (sketch).

III. Some Multiparty Bell inequalities.

I. The Tsirelson bound(s).

First we look at the CHSH scenario and reformulate it here in matrix (or operator) language.

Recall Alice and Bob make measurements of the observables A, A', B, B' where

$$A = (+1)\,|\alpha\rangle\langle\alpha| + (-1)\,|\alpha_\perp\rangle\langle\alpha_\perp|$$
$$A' = (+1)\,|\alpha'\rangle\langle\alpha'| + (-1)\,|\alpha'_\perp\rangle\langle\alpha'_\perp|$$

$$B = (+1)\,|\beta\rangle\langle\beta| + (-1)\,|\beta_\perp\rangle\langle\beta_\perp|$$

$$B' = (+1)\,|\beta'\rangle\langle\beta'| + (-1)\,|\beta'_\perp\rangle\langle\beta'_\perp| .$$

It is easy (exercise) to see that for any

$|\gamma\rangle = \cos\gamma\,|0\rangle + \sin\gamma\,|1\rangle$; $|\gamma_\perp\rangle = \sin\gamma\,|0\rangle - \cos\gamma\,|1\rangle$

we have

$$|\gamma\rangle\langle\gamma| - |\gamma_\perp\rangle\langle\gamma_\perp| = \begin{pmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{pmatrix}$$

$$= (\cos 2\gamma)\,\sigma_z + (\sin 2\gamma)\,\sigma_x$$

$$= (\sin 2\gamma, 0, \cos 2\gamma) \cdot \underbrace{\vec{\sigma}}$$

$$(\sigma_x, \sigma_y, \sigma_z) = \vec{\sigma} .$$

So measuring this observable means measuring the "spin" in the direction $(\sin 2\gamma, 0, \cos 2\gamma)$.

Possible results of the meas are ± 1 corresponding to states $|\gamma\rangle$ and $|\gamma_\perp\rangle$.

We also introduce the "Bell operator"

$$\mathcal{B} = A \otimes B - A \otimes B' + A' \otimes B + A' \otimes B'$$

which is here a 4×4 matrix acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The correlation coefficient introduced in the Bell experiment is thus $\langle \psi | \mathcal{B} | \psi \rangle$. We have seen that for $|\psi\rangle = |\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and the right choice of $\alpha, \beta, \alpha', \beta'$ (so of A, B, A', B') we get

$\langle \psi | \mathcal{B} | \psi \rangle = 2\sqrt{2}$. Is this the maximal possible value? The answer is yes. We reformulate this question slightly and prove that this is the max possible value in QM.

Theorem: Let A, B, A', B' be 2×2 hermitian matrices acting on $\mathbb{C}^2$ (for A, A') and a copy of $\mathbb{C}^2$ (for B, B') such that

$$(*) \quad A^2 = A'^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^2 = B'^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let
$$T_{\text{tensor}} = \max_{\substack{|\psi\rangle, A, B, A', B' \\ \text{such that } (*) \text{ holds}}} \langle \psi | \mathcal{B} | \psi \rangle.$$

Then we have $T_{\text{tensor}} = 2\sqrt{2}$.

Proof.

We first observe that by Cauchy-Schwarz:

$$\langle \psi | \mathcal{B} | \psi \rangle^2 \leq \underbrace{\|\psi\|^2}_{1} \underbrace{\|\mathcal{B}\psi\|^2}$$
$$= \langle \psi | \mathcal{B}^2 | \psi \rangle$$

So we study the operator $\mathcal{B}^2$.

Using $A^2 = A'^2 = \mathbb{1}$, $B^2 = B'^2 = \mathbb{1}$ we can show by pure algebra that

$$\mathcal{B}^2 = 4\, \mathbb{1} \otimes \mathbb{1} - [A, A'] \otimes [B, B']$$

where $[M, N] = MN - NM$ is called the "commutator".

Now, $\mathcal{B}^2$ is a hermitian matrix so we have that $\langle \psi | \mathcal{B}^2 | \psi \rangle \leq \underbrace{\lambda_{max}}(\mathcal{B}^2)$

the maximal eigenvalue of $\mathcal{B}^2$.
This is also a norm $\|\mathcal{B}^2\|_{op}$.

By the triangle inequality for a norm

$$\|\mathcal{B}^2\|_{op} \leq 4 + \|[A, A'] \otimes [B, B']\|_{op}.$$

$$= 4 + \|[A, A']\|_{op} \|[B, B']\|_{op}.$$

$$\leq 4 + (\|AA'\| + \|A'A\|)(\|BB'\| + \|B'B\|_{op})$$

$$\leq 4 + 4 \|A\| \|A'\| \|B\| \|B'\|.$$

All norms $\|A\|_{op} \ldots$ are equal to $\underbrace{\lambda_{max}(A)\ldots}_{+1.}$

$$\Rightarrow \quad \|\mathcal{B}^2\|_{op} \leq 4+4 = 8.$$

$$\Rightarrow \quad \langle\psi|\mathcal{B}^2|\psi\rangle \leq 8$$

$$\Rightarrow \quad \langle\psi|\mathcal{B}|\psi\rangle^2 \leq 8 \Rightarrow \boxed{|\langle\psi|\mathcal{B}|\psi\rangle| \leq 2\sqrt{2}}$$

Finally this maximal value is attained for $|\psi\rangle = |\psi^+\rangle$ and a special choice of A, A', B, B' (with right angles $\alpha, \alpha', \beta, \beta'$).

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<u>Remark:</u> The Tsirelson identity

$$\mathcal{B}^2 = 4\, \mathbb{1} \otimes \mathbb{1} - [A, A'] \otimes [B, B']$$

is very suggestive. First it "shows" that in a "classical" (i.e non quantum) setting where the algebra of observables is a commuting algebra we would have $[A, A'] = 0$ and/or $[B, B'] = 0$ so that $\mathcal{B}^2 = 4\, \mathbb{1} \otimes \mathbb{1}$ which implies then $|\langle \psi | \mathcal{B} | \psi \rangle| \leq 2$. (by Cauchy-Schwarz again).

Thus to violate the Bell-CHSH ineqn we need non-commutative algebras of observables.

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As we will also see later this kind of "identities" allow to guess generalized Bell inequalities.

II. Tsirelson's problem.

Let us now turn to "Tsirelson's problem".

The computation of T_{tensor} is an optimization problem. We can consider a slightly more general problem where $\mathbb{C}^2 \otimes \mathbb{C}^2$ is replaced by $\mathbb{C}^d \otimes \mathbb{C}^d$ (so the local Hilbert space of Alice and Bob is $\mathbb{C}^d$) with $d \geq 2$. The matrices A, A', B, B' are $\underline{d \times d}$; they still satisfy the algebraic condition $A^2 = A'^2 = \mathbb{1}_{d\times d}$, $B^2 = B'^2 = \mathbb{1}_{d\times d}$ and are hermitian.

$$T_{tensor}(d) = \max_{\substack{|\psi\rangle \\ A, A', B, B'}} \langle \psi | \mathcal{B} | \psi \rangle$$

with $\mathcal{B} = A \otimes B - A \otimes B' + A' \otimes B + A' \otimes B'$.

Tsirelson then proceeded to define a variation of this optimization problem as follows. Let $\mathcal{H} = \mathbb{C}^{d^2}$ the Hilbert space of Alice & Bob (of course isomorphic to $\mathbb{C}^d \otimes \mathbb{C}^d$ as long as $d < \infty$) and consider the algebra of hermitian $\underline{\underline{d^2 \times d^2}}$ matrices A, A', B, B' such that

$$\begin{cases} A^2 = A'^2 = B^2 = B'^2 = \mathbb{1}_{d^2 \times d^2} \\ [A, B] = [A', B'] = 0. \end{cases}$$

$$T_{\text{Commute}} \equiv \max_{\substack{|\psi\rangle \\ A, A', B, B'}} \langle \psi | AB - AB' + A'B + A'B' | \psi \rangle$$

<u>Theorem</u>: as long as d is finite. we have

$$T_{\text{tensor}} = T_{\text{commute}}.$$

Of course we have that in the first formulation with A & B $d \times d$ matrices (ect for the others)

$$A \otimes B = (A \otimes \mathbb{1})(\mathbb{1} \otimes B)$$

and $\left[A \otimes \mathbb{1}, \mathbb{1} \otimes B \right] = 0$.

So it should be clear that the second optimization problem maximizes over a potentially larger space of matrices so that $T_{\text{tensor}} \leq T_{\text{commute}}$.

In finite dimensions it turns out that the space of commuting $d^2 \times d^2$ $\overbrace{\text{matrices with}}^{A \,\&\, B}$ $[A, B] = 0$ is "isomorphic" to $A \otimes \mathbb{1}$, $\mathbb{1} \otimes B$ (for $d \times d$ matrices here).

and therefore the <u>two optimization problems</u> <u>are equivalent</u>: $\boxed{T_{\text{tensor}} = T_{\text{commute}}}$

<u>Tsirelson's problem.</u>: Prove or disprove that

this is also the case for $d = +\infty$. Of course we have not even defined the problem precisely for infinite dimensional spaces as we should define infinite dimensional Hilbert-spaces, their tensor products in infinite dimensions, algebras of observables in infinite dimensions ect...

The Tsirelson problem has been solved in the negative recently (so $T_t \neq T_c$ in infinite dimensions) and it has turned out that the math is deeply related to other famous conjectures in operator algebras and computer science. These bridges between different conjectures in different fields where quite unexpected.

For a (computer science oriented) "popular" account see:
Hartnett Kevin Quanta Magazine (4 March 2020)
"Landmark Computer Science Proof Cascades Through Physics and Math"

III. Multiparty Bell inequalities.

There exist a whole zoology of Bell-type inequalities nowadays. The subject has expanded considerably and it has turned out that the various manifestation of entanglement for multiparty systems is far from understood and incredibly rich,

These inequalities can be generalized in various directions. For example one may consider n parties (Alice, Bob, Charlie, ...) that each can measure observables in m settings (Alice has $A, A', A'', \ldots$ Bob has $B, B', B'', \ldots$ Charlie has $C, C', C'', \ldots$) and also the dimension of the local Hilbert space can be $d \geq 2$.

For the usual CHSH we have $n=2$, $m=2$, $d=2$

and we speak about a $BI(2, 2, 2)$. In more general situations as described above we speak of a $BI(m, m, d)$. Of course we can imagine even more general situations.

Here we will take some inspiration from the method of proof of Tsirelson for $T_{tensor}(2, 2, 2) = 2\sqrt{2}$ to uncover new Bell inequalities.

We only discuss here $BI(3, 2, 2)$ but it will somehow be clear that the discussion generalizes to $BI(m, 2, 2)$. (Often these are also called Mermin inequalities as $BI(3, 2, 2)$ is equivalent to a proposal of D. Mermin).

Informal discussion.

We have $n = 3$ so Alice, Bob, Charlie (3 parties). Each has two possible measurement settings (or devices) described by observables A, A', B, B', C, C' which are all hermitian 2×2 matrices. We assume $A^2 = A'^2 = \mathbb{1}_{2\times 2}$, $B^2 = B'^2 = \mathbb{1}_{2\times 2}$, $C^2 = C'^2 = \mathbb{1}_{2\times 2}$

Taking inspiration from the CHSH case we construct an operator $\mathcal{C}$ such that (analog of $\mathcal{B}$):

$$\mathcal{C}^2 = 4\, \mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1} - \{ [A, A']\otimes[B, B']\otimes\mathbb{1} + [A, A']\otimes\mathbb{1}\otimes[C, C'] + \mathbb{1}\otimes[B, B']\otimes[C, C'] \}.$$

The operator (matrix) $\mathcal{C}$ is the "square-root" and will be given later.

For "classical" observables that would commute

we would have $[A, A'] = [B, B'] = [C, C'] = 0$

so $\mathcal{C}^2 = 4\, \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$ and thus:

$$|\mathcal{C}_{\text{classical}}| \leq 2.$$

This is a Bell-type inequality. (analog of $|\mathcal{B}| \leq 2$)

For the quantum case where observables do not commute (given here by 2×2 matrices) since $A^2 = \mathbb{1}$ (idem for the rest) their eigenvalues are ± 1 (just as for CHSH). Therefore we see that:

$$\|\mathcal{C}^2\|_{op} \leq 4 + \{2 \times 2 + 2 \times 2 + 2 \times 2\}$$
$$= 16. \quad \text{(similar bounds as for } \mathcal{B}^2\text{)}$$

$$\Rightarrow |\langle\psi|\mathcal{C}|\psi\rangle|^2 \leq \langle\psi|\mathcal{C}^2|\psi\rangle \leq 16$$

$$\Rightarrow \quad |\langle \psi | \mathcal{C} | \psi \rangle| \leq 4 .$$

Thus we expect a maximal violation of the "Bell inequality" $|\mathcal{C}_{QM}| = 4$ if we can find appropriate $|\psi\rangle$ & A, B, C, A', B', C'.

One can check that the $\mathcal{C}$ that leads to the $\mathcal{C}^2$ (above on page 15) is simply:

$$\boxed{\mathcal{C} = - A \otimes B \otimes C + A \otimes B' \otimes C' + A' \otimes B \otimes C' + A' \otimes B' \otimes C}$$

and the $|\psi\rangle$ that yields max violation for appropriate choice of angle $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ (or observables) is

$$\boxed{|\psi\rangle = |GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)}.$$

the Greenberger-Horne-Zeilinger state which generalizes the EPR pair or Bell state.

In other words

$$\boxed{T_{\text{tensor}} = \max_{\substack{|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \\ A, B, C \text{ s.t. the} \\ A', B', C' \text{ squares} \\ = \mathbb{1}_{2\times 2}}} \langle \psi | \mathcal{C} | \psi \rangle = 4.}$$

↑ Tsirelson bound

In conclusion let us just say that this construction can be generalized to any m:

$BI(m,2,2)$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$ ...
classical bound	2	2	4	4	8 ...
quantum Tsirelson bound	$2\sqrt{2}$	4	$8\sqrt{2}$	16	$32\sqrt{2}$ ...
Ratio $R = \frac{QM}{\text{classical}}$	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$.

The states that saturate these bounds are

$$|\psi\rangle = \frac{1}{\sqrt{2}} \Big(\underbrace{|00\cdots0\rangle}_{m\text{-fold tensor product}} + |11\cdots1\rangle \Big).$$